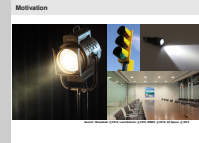# Area-preserving parameterizations for spherical ellipses

Ibón Guillén[1,2]   Carlos Ureña[3]   Alan King[4]   Marcos Fajardo[4]
Iliyan Georgiev[4]   Jorge López-Moreno[2]   Adrian Jarabo[1]

[1]Universidad de Zaragoza, I3A
[2]Universidad Rey Juan Carlos
[3]Universidad de Granada
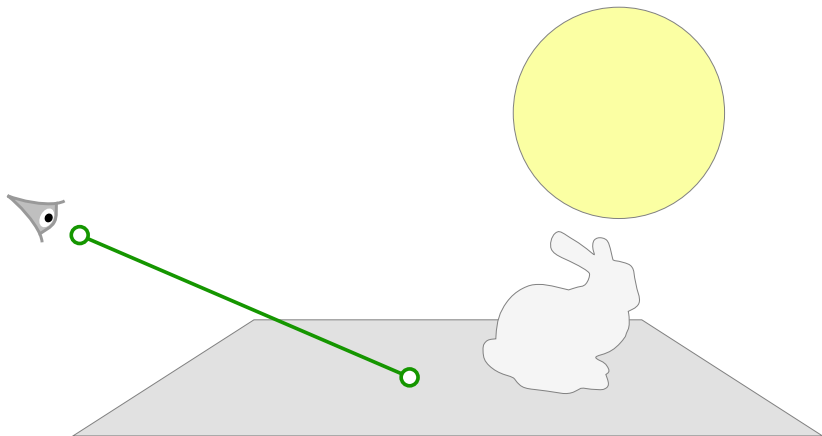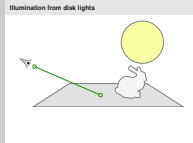[4]Solid Angle

Thank you for the introduction

Motivation

Source: WiseGeek ©2016, Luca Sartoni ©2010, NEBO ©2016, K2 Space ©2012

Disk-shaped luminaries are ubiquitous and present in lots of everyday scenes.

Spotlights, traffic lights, flash-lights or ceil lights are some of the examples of luminaries whose basic shape resembles a disk.
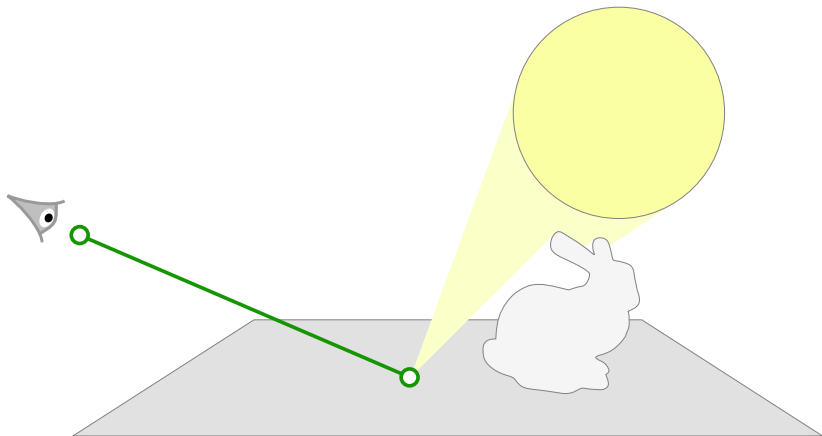
Illumination from disk lights

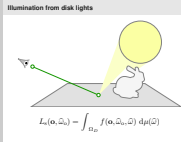We want to compute the incident radiance at a given surface point from a disk-shaped luminaire.
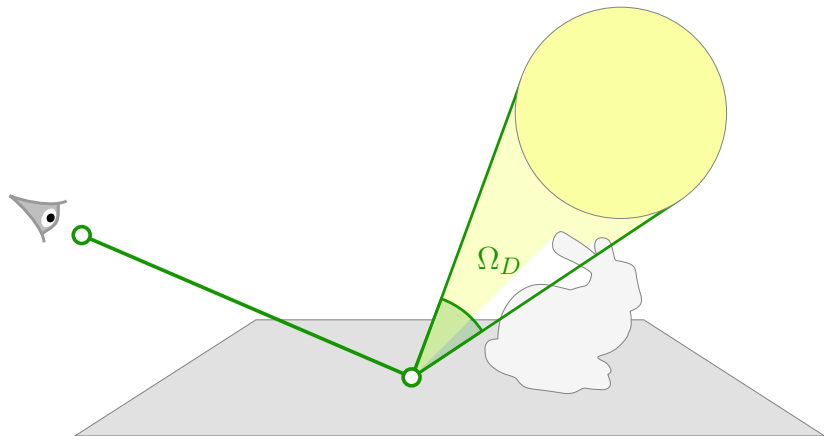
1

Illumination from disk lights

$$L_\mathrm{s}(\mathbf{o}, \widehat{\omega}_\mathrm{o}) = \int_{\Omega_D} f(\mathbf{o}, \widehat{\omega}_\mathrm{o}, \widehat{\omega}) \ \mathrm{d}\mu(\widehat{\omega})$$
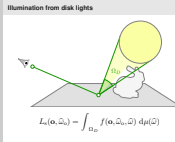
Here *f* is the contribution function, which takes into account the emission profile, the medium attenuation and/or the geometric foreshortening. This function gives the irradiance arriving from a given direction.

1

Illumination from disk lights

$$L_\text{s}(\mathbf{o}, \widehat{\omega}_\text{o}) = \int_{\Omega_D} f(\mathbf{o}, \widehat{\omega}_\text{o}, \widehat{\omega}) \ \mathrm{d}\mu(\widehat{\omega})$$

We need to integrate *f* over the set of directions Ω_D (solid angle) subtended by the luminarie.
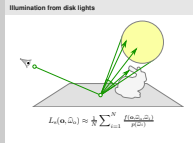
Since in most cases this integral does not have a closed form, we rely on numerical methods, commonly Monte Carlo integration for computing this integral.

1

Illumination from disk lights

$$L_{\mathrm{s}}(\mathbf{o}, \widehat{\omega}_{\mathrm{o}}) \approx \tfrac{1}{N} \sum_{i=1}^{N} \frac{f(\mathbf{o}, \widehat{\omega}_{\mathrm{o}}, \widehat{\omega}_i)}{p(\widehat{\omega}_i)}$$

For this purpose, we would need to generate some samples, which we average. However, how this samples are generated is crucial for reducing variance, and we would like to generate such samples so that they are as approximated as possible to the function being integrated.
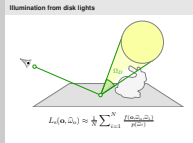
In particular, what we would like is to compute such samples uniformly with respect to the solid angle Ω_D

$$L_{\mathrm{s}}(\mathbf{o}, \widehat{\omega}_{\mathrm{o}}) \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(\mathbf{o}, \widehat{\omega}_{\mathrm{o}}, \widehat{\omega}_{i})}{p(\widehat{\omega}_{i})}$$

Ω_D

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In particular, what we would like is to compute such samples uniformly with respect to the solid angle Ω_D

Area sampling
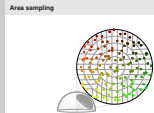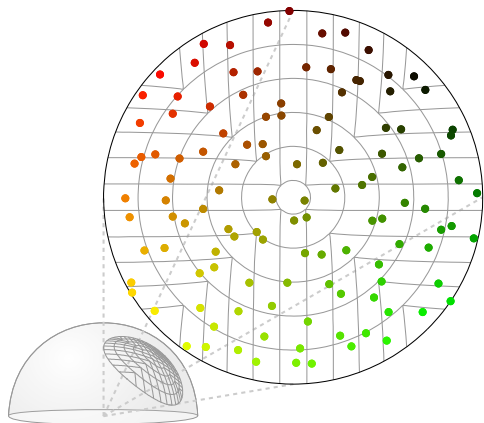
The most straightforward method is to generate samples over the surface of the disk.
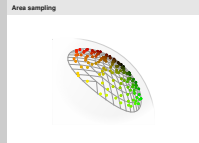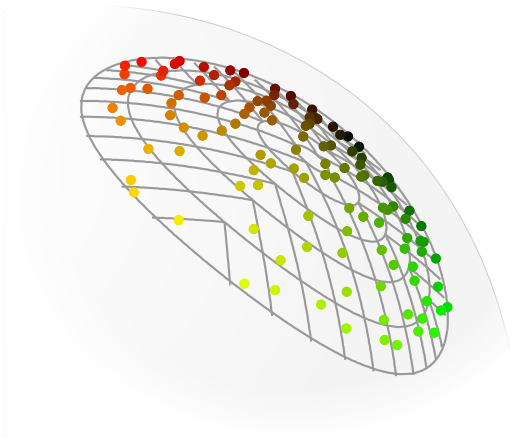
However, this results in an distribution of sampling directions which are not proportional to the sustained solid angle, and even worse, it introduces singularities in the integrand.

Area-preserving parameterizations for spherical ellipses
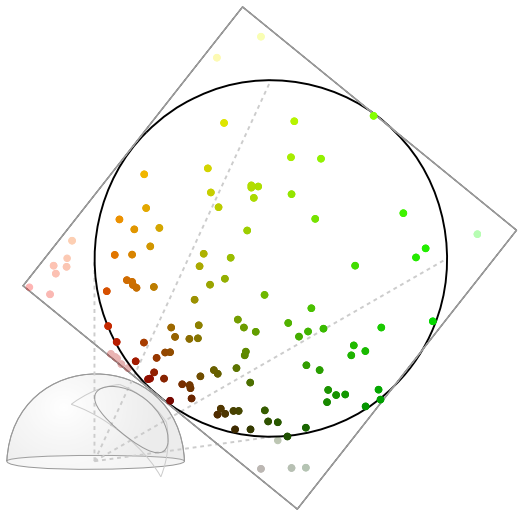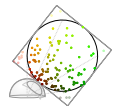
Recently, Gamito proposed to generate samples uniformly in the disk's solid angle, by sampling the quad bounding the disk using Ureña's method , and rejecting those which fall outside the disk.

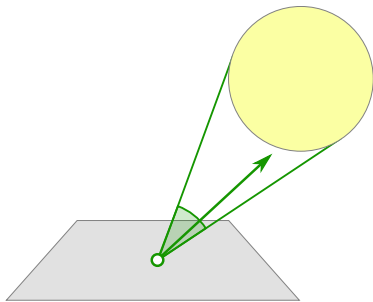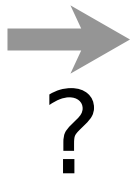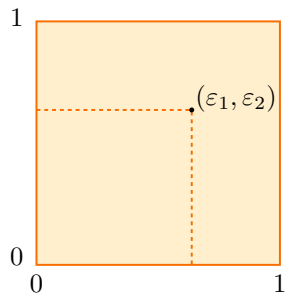Area-preserving parameterizations for spherical ellipses
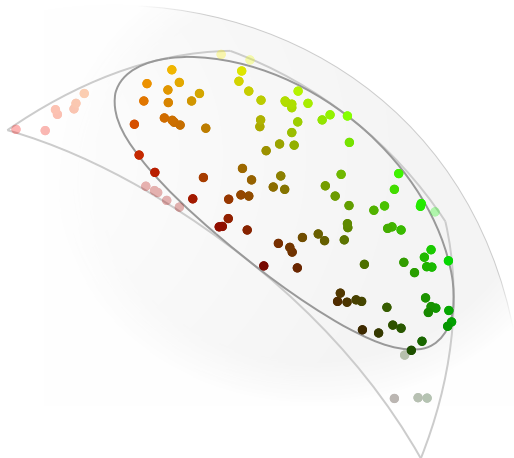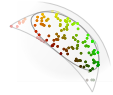
However, due to the samples rejection, the sampling pattern cannot have a fixed size.

Therefore, the method does not preserve stratification of the samples in the solid angle, or across sampling dimensions.
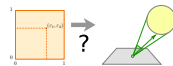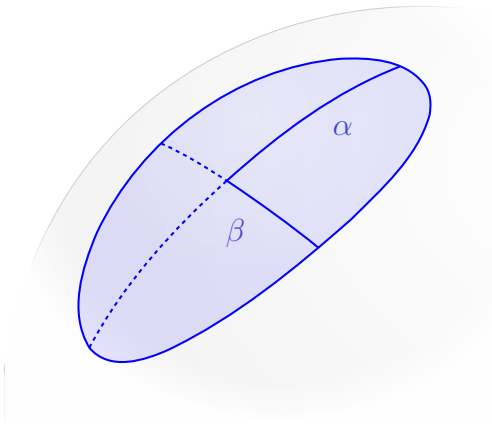
Goal

- To provide a **direct mapping** from the unit square to directions contained on the disk's subtended solid angle.

- It's **continuous** on the entire domain.

- It preserves **fractional area**.

$(\varepsilon_1, \varepsilon_2)$

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Spherical ellipse

Key Observation: The perspective projection of a disk[1] in a sphere is a spherical ellipse.


Every possible disk (and more general quadrics like ellipses or ellipsoids), when projected into the sphere, results in a spherical ellipse, which is defined its semi-major arc α and a semi-minor arc β.

So, we are no longer on the domain of the disk, but on the domain of the spherical ellipse.
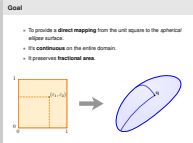
Details about how to obtain the projection parameters in the paper.

Goal

- To provide a **direct mapping** from the unit square to the *spherical ellipse* surface.

- It's **continuous** on the entire domain.

- It preserves **fractional area**.
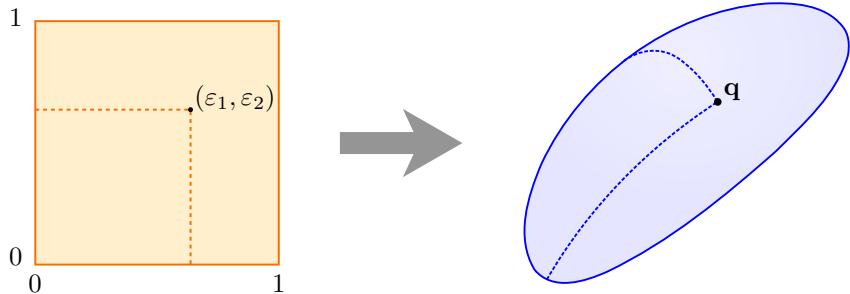
Therefore, our goal is to provide a direct mapping over the spherical ellipse's surface.

8

- To provide a **direct mapping** from the unit square to the spherical ellipse surface.
- It's **continuous** on the entire domain.
- It preserves **fractional area**.

Goal

- To provide a **direct mapping** from the unit square to the *spherical ellipse* surface.

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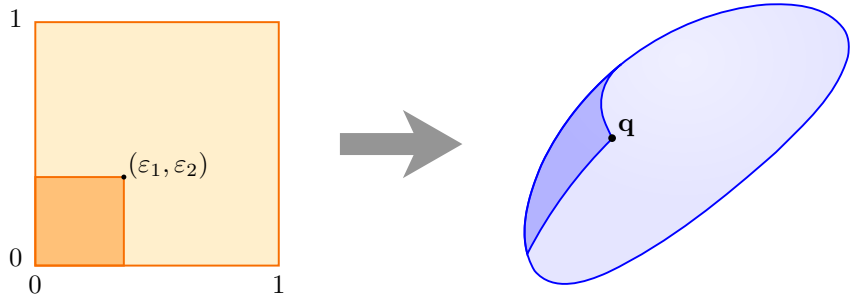
- It preserves **fractional area**.
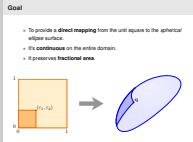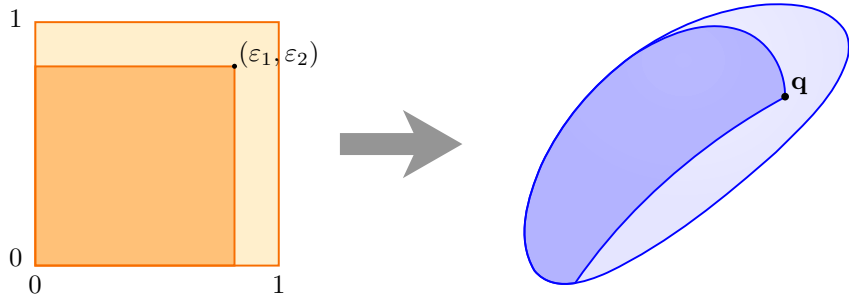

Area-preserving parameterizations for spherical ellipses

Being area-preserving means that our map translates regions of certain area in the unit square to regions on the spherical ellipse maintaining the proportionality with respect to the complete surface area.

9

Goal

- To provide a **direct mapping** from the unit square to the spherical ellipse surface.
- It's **continuous** on the entire domain.
- It preserves **fractional area**.

Goal

- To provide a **direct mapping** from the unit square to the *spherical ellipse* surface.

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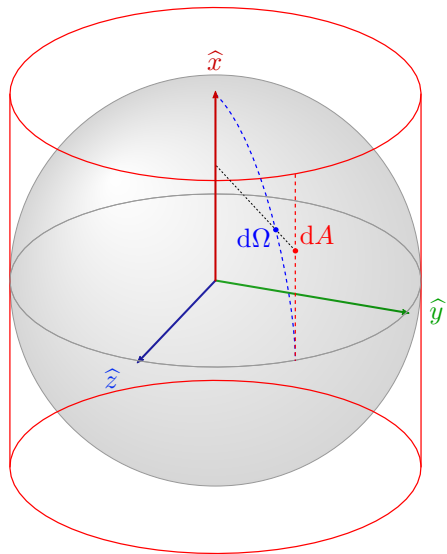- It preserves **fractional area**.

Area-preserving parameterizations for spherical ellipses
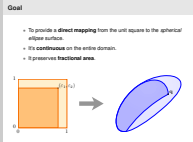
Unfortunately, finding this mapping is not trivial. For that, we will use Archimedes Box-Hat theorem, which as we will see will allow to simplify the problem.
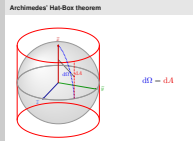
In the following, we will introduce it, and show how to use to define our two area preserving mappings.

10

$$\mathrm{d}\Omega = \mathrm{d}A$$
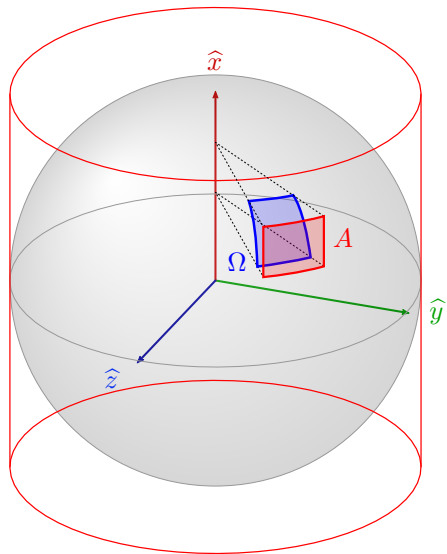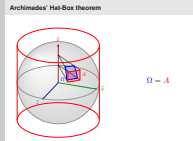
Following the Hat-Box theorem, any differential of area on the unit sphere (blue) can be radially projected to a corresponding differential of area (red) on a surrounding cylinder.

# Archimedes' Hat-Box theorem

$$\Omega = A$$

The Hat-Box theorem does not apply only to differential areas, but any region Ω (blue) on the unit sphere can be radially projected to another region A (red) on a surrounding cylinder

The mapping preserves the areas of both regions.

Thus, to obtain a point on the spherical region, we can uniformly sample inside the cylindrical region and project back onto the sphere.

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# Our mappings
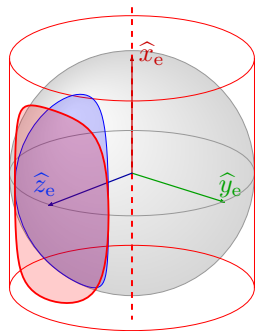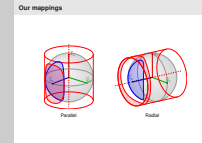
Parallel
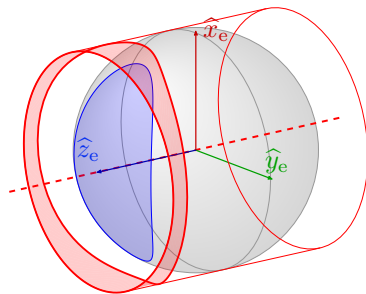
Radial

The Box-Hat theorem holds for a bounding cylinder in any orientation.

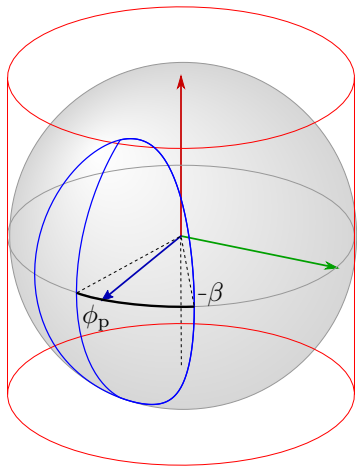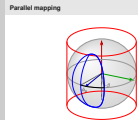Concretely, we are going to work with two particular cylindrical orientations:

- Aligned with the spherical ellipse's $\widehat{x}_e$, which is the canonical disposition and is the basis of our parallel mapping.

- And also another aligned with the spherical ellipse's $\widehat{z}_e$, from which we'll develop our radial mapping.
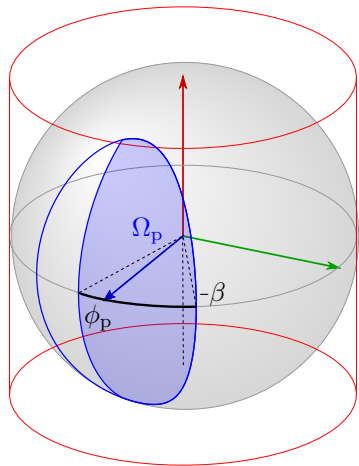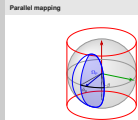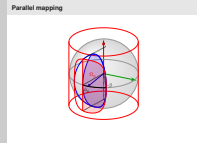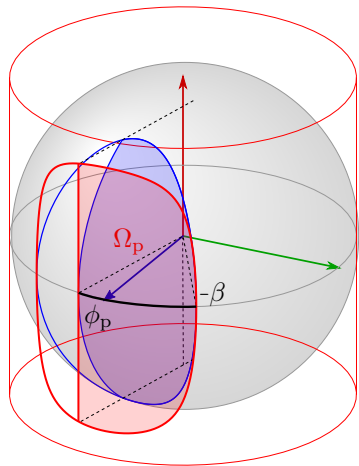
Our mappings
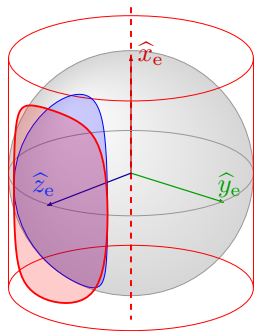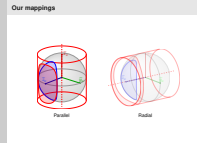
Parallel
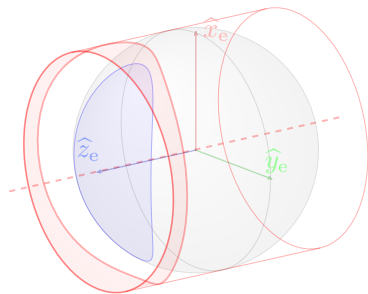
Radial

So, we have two sampling domains: the first one defined as this angle phi, while the second is uniformly sampled along this blue arc.
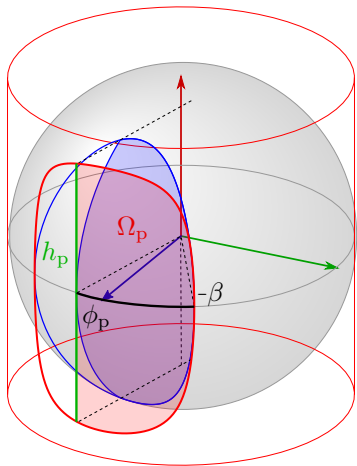
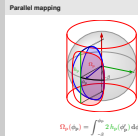We need to characterize the partial spherical surface covered by this angle.

16

And for that purpose we use the Hat-Box theorem and we compute the area of this region in the bounding cylinder.

Parallel mapping

$$\Omega_{\mathrm{p}}(\phi_{\mathrm{p}}) = \int_{-\beta}^{\phi_{\mathrm{p}}} 2\,h_{\mathrm{p}}(\phi'_{\mathrm{p}})\,\mathrm{d}\phi'_{\mathrm{p}}$$
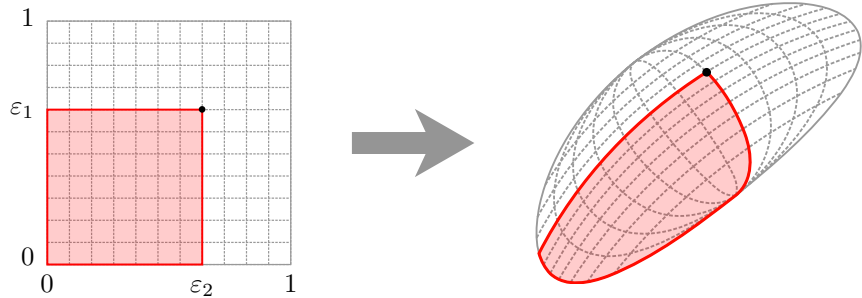
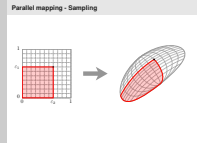Area-preserving parameterizations for spherical ellipses

Which in the end means we have to integrate the line segments resulting of the projection of each arc.

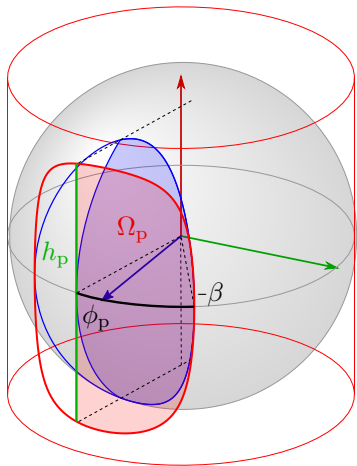The second sampling domain can then sampled uniformly along the segment.

This is the resulting mapping between the unit square and the spherical ellipse.

$$\Omega_{\mathrm{p}}(\phi_{\mathrm{p}}) = \int_{-\beta}^{\phi_{\mathrm{p}}} 2\, h_{\mathrm{p}}(\phi_{\mathrm{p}}')\, \mathrm{d}\phi_{\mathrm{p}}'$$
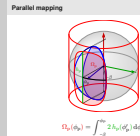
We need an analytic expression to calculate this integral...

Which after transformations results in the following analytical expression.

This equation unfortunately contains two elliptical integrals, one of the third kind and one of the first kind. We will like to have something simpler.

$$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \tfrac{2c_{\mathrm{t}}}{b_{\mathrm{t}}}\left[(1-n)\,\Pi\left(n;\varphi_{\mathrm{p}}|m\right) - \mathrm{F}\left(\varphi_{\mathrm{p}}|m\right)\right]$$

Area-preserving parameterizations for spherical ellipses

$$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \tfrac{2c_{\mathrm{t}}}{b_{\mathrm{t}}} \left[(1-n) \underbrace{\Pi\left(n; \varphi_{\mathrm{p}}|m\right)}_{\text{Third kind}} - \underbrace{\mathrm{F}\left(\varphi_{\mathrm{p}}|m\right)}_{\text{First kind}}\right]$$

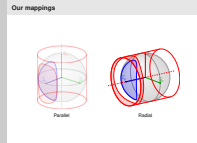Which after transformations results in the following analytical expression.

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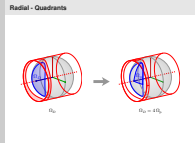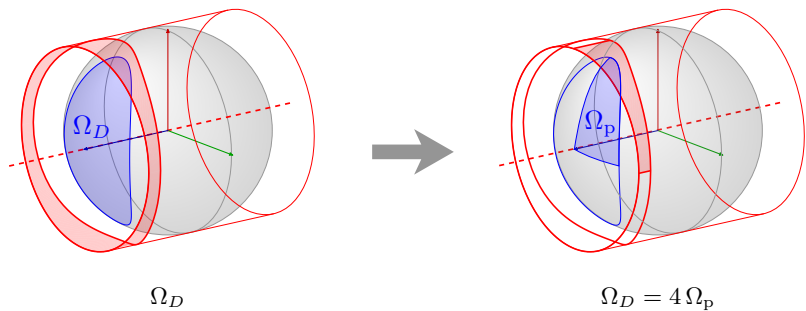
$$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \tfrac{2c_{\mathrm{t}}}{b_{\mathrm{t}}} \left[(1-n) \underbrace{\Pi\left(n; \varphi_{\mathrm{p}}|m\right)}_{\text{Third kind}} - \underbrace{\mathrm{F}\left(\varphi_{\mathrm{p}}|m\right)}_{\text{First kind}}\right]$$

Our mappings
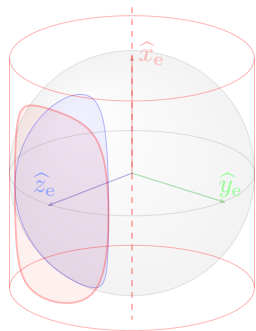
Parallel
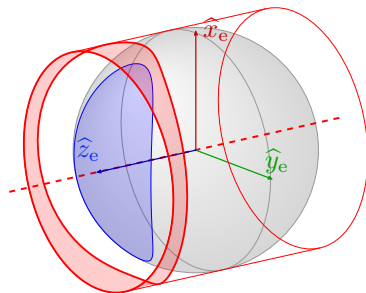
Radial

Now we start from the $\widehat{z}_e$ aligned radial projection.

Radial - Quadrants

Ω_D

$\Omega_D = 4\,\Omega_\mathrm{p}$

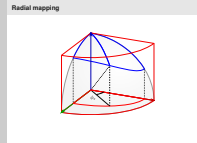Area-preserving parameterizations for spherical ellipses

By exploiting the symmetry between the four quadrants of the spherical ellipse, we are going to reduce the problem to uniformly sampling the first quadrant, and later we'll retranslate the samples to the remaining quadrants.
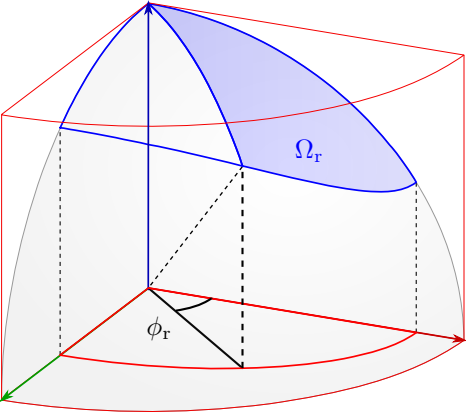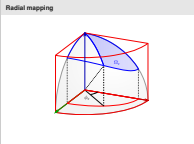
23
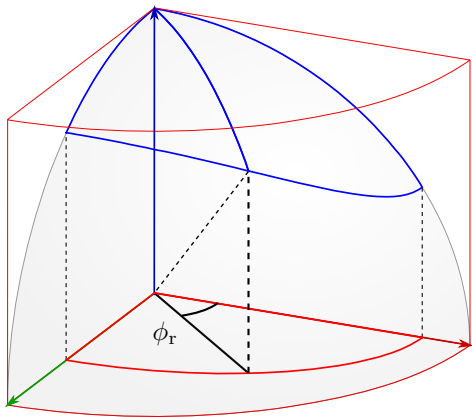
Radial mapping

Now our first sampling dimension is the radial angle, and the second is uniformly sampled along the blue arc.

ϕ_r

Radial mapping
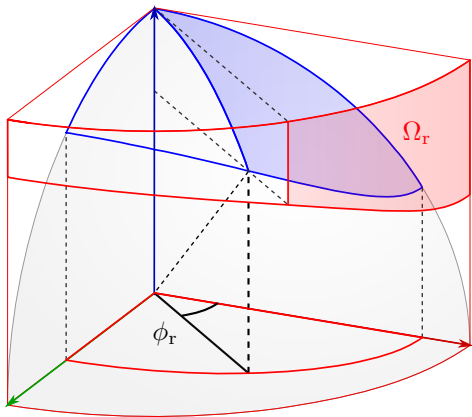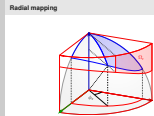
We need to characterize the partial spherical surface covered by this angle.

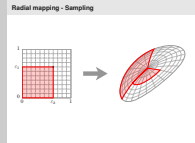Ω_r

ϕ_r

And for that purpose we use the Box-HAx theorem and we compute the area of this region into the bounding cylinder.

Note that in this case we are actually integrating the complementary of the area.

$\Omega_{\rm r}$

$h_{\rm r}$

$\phi_{\rm r}$

$$\Omega_{\rm r}(\phi_{\rm r}) = \phi_{\rm r} - \int_0^{\phi_{\rm r}} h_{\rm r}(\phi_{\rm r}') \, {\rm d}\phi_{\rm r}'$$
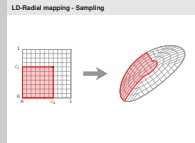
Radial mapping - Sampling
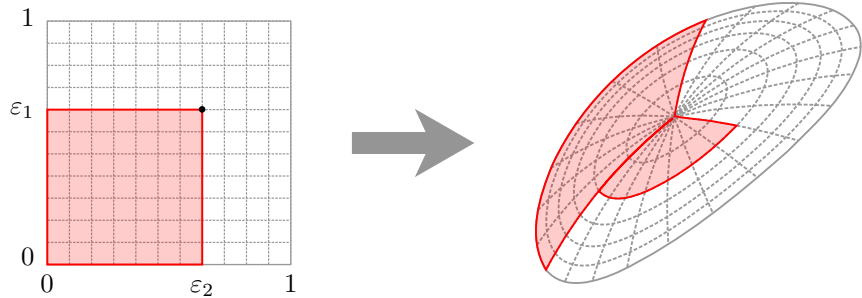
This is the resulting map. Note that retranslating the samples to the remaining quadrants introduces a singularity into the map.
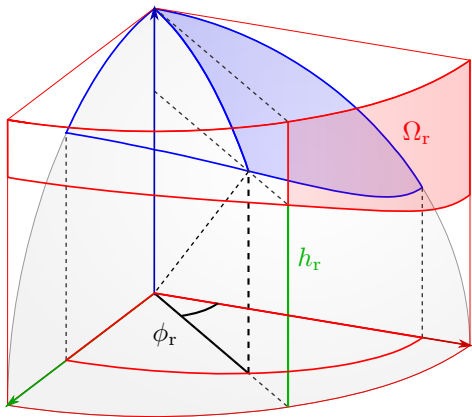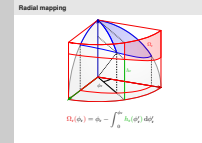
28
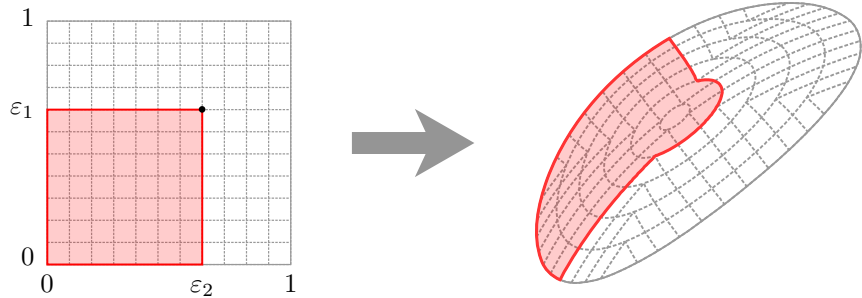
We redistribute the samples following the technique of Shirley and Chiu [1997] for circle polar mapping before applying our map.

Again, we want to find a close form for this expression.

$$\Omega_{\mathrm{r}}(\phi_{\mathrm{r}}) = \phi_{\mathrm{r}} - \int_{0}^{\phi_{\mathrm{r}}} h_{\mathrm{r}}(\phi'_{\mathrm{r}}) \, \mathrm{d}\phi'_{\mathrm{r}}$$

Area-preserving parameterizations for spherical ellipses

The resulting expression contains a single elliptical integral of the third kind, as opposed to the the previous form.

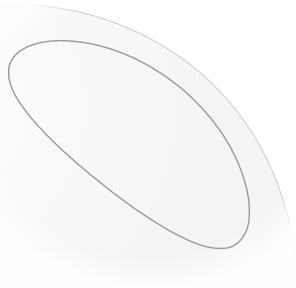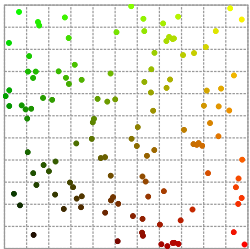$$\Omega_{\rm r}(\phi_{\rm r}) = \phi_{\rm r} - \frac{b(1-a^2)}{a\sqrt{1-b^2}}\,\Pi\,(n;\varphi_{\rm r}|m)$$

The resulting expression contains a single elliptical integral of the third kind, as opposed to the the previous form.

$$\Omega_{\rm r}(\phi_{\rm r}) = \phi_{\rm r} - \frac{b(1-a^2)}{a\sqrt{1-b^2}} \underbrace{\Pi\left(n; \varphi_{\rm r}|m\right)}_{\text{Third kind}}$$

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Sampling comparison
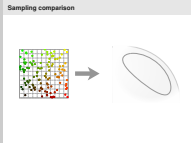
Now, let's take a look at the resulting maps, from these sample points in the unit square to the spherical ellipse.
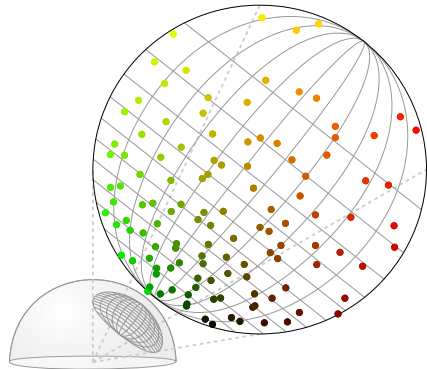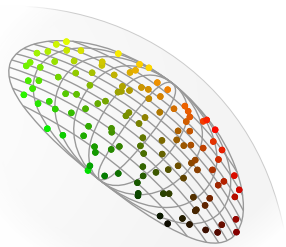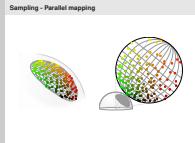
This is he first parallel mapping. Note that all samples are uniformly distributed in the spherical ellipse, while the distrinution is not uniform on the disc on the right.

Sampling - Radial

Our radial mappin with a discontinuity.

Area-preserving parameterizations for spherical ellipses
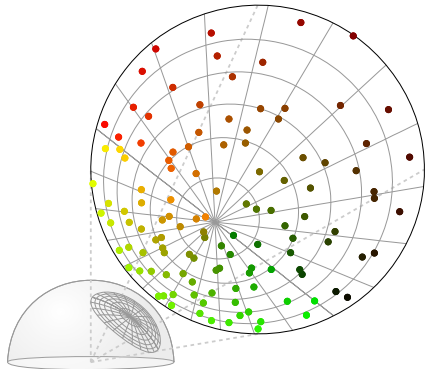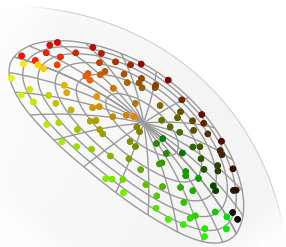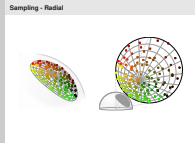

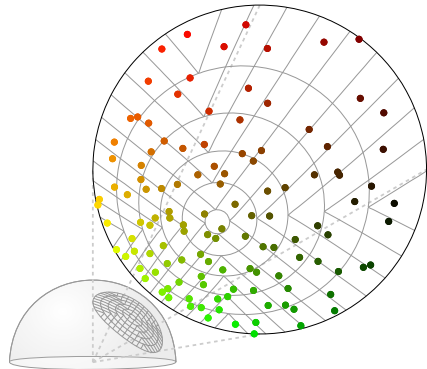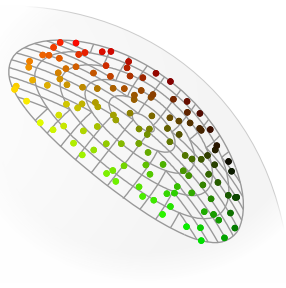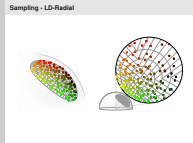Sampling - LD-Radial

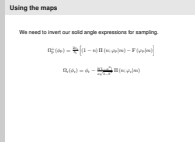And our continuous radial mapping.

So far...

- Three area-preserving mappings for spherical ellipses, making use of the Box-Hat theorem.

- **Collateral:** Novel expressions for the disk solid angle

  - Simpler than previous approaches [Paxton 1959, Conway 2009], involving only one/two elliptical integral

Using the maps

We need to invert our solid angle expressions for sampling.

$$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \frac{2c_{\mathrm{t}}}{b_{\mathrm{t}}} \left[(1-n)\,\Pi\left(n; \varphi_{\mathrm{p}}|m\right) - \mathrm{F}\left(\varphi_{\mathrm{p}}|m\right) \right]$$

$$\Omega_{\mathrm{r}}(\phi_{\mathrm{r}}) = \phi_{\mathrm{r}} - \frac{b(1-a^2)}{a\sqrt{1-b^2}}\,\Pi\left(n; \varphi_{\mathrm{r}}|m\right)$$

The lack of a closed form forces us to result to numerical methods to generate the samples.

We resort to tabulation.

Using the maps

We need to invert our solid angle expressions for sampling.

$$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \frac{2c_{\mathrm{t}}}{b_{\mathrm{t}}} \left[(1-n) \, \Pi \left(n; \varphi_{\mathrm{p}} | m \right) - \mathrm{F} \left(\varphi_{\mathrm{p}} | m \right) \right]$$

$$\Omega_{\mathrm{r}}(\phi_{\mathrm{r}}) = \phi_{\mathrm{r}} - \frac{b(1-a^2)}{a\sqrt{1-b^2}} \, \Pi \left(n; \varphi_{\mathrm{r}} | m \right)$$

No closed form! $\rightarrow$ Need numerical inversion for sampling generation.

$$\Omega(\phi) - \varepsilon_1 \Omega_D = 0.$$

Might become a bottleneck! $\rightarrow$ Tabulation

Using the maps

We need to invert our solid angle expressions for sampling.

$\Omega_{\mathrm{p}}^{+}(\phi_{\mathrm{p}}) = \frac{2c_{\mathrm{t}}}{b_{\mathrm{t}}} \left[(1-n) \, \Pi \left(n; \varphi_{\mathrm{p}} | m \right) - \mathrm{F} \left(\varphi_{\mathrm{p}} | m \right) \right]$

$\Omega_{\mathrm{r}}(\phi_{\mathrm{r}}) = \phi_{\mathrm{r}} - \frac{b(1-a^2)}{a\sqrt{1-b^2}} \, \Pi \left(n; \varphi_{\mathrm{r}} | m \right)$

No closed form! $\rightarrow$ Need numerical inversion for sampling generation.

$\Omega(\phi) - \varepsilon_1 \Omega_D = 0.$

Might become a bottleneck! $\rightarrow$ Tabulation

Area-preserving parameterizations for spherical ellipses

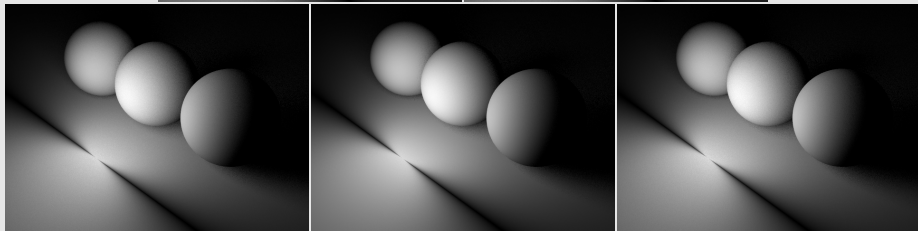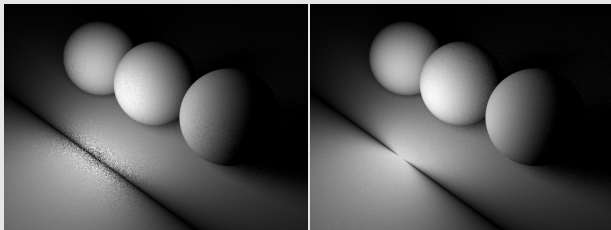The lack of a closed form forces us to result to numerical methods to generate the samples.

We resort to tabulation.

And now we are going to see some results of using our maps to sample disk-shaped luminaries inside a Monte Carlo renderer.

Mitsuba

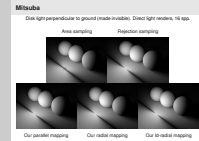Disk light perpendicular to ground (made invisible). Direct light renders, 16 spp.

Area sampling   Rejection sampling

Our parallel mapping   Our radial mapping   Our ld-radial mapping

39
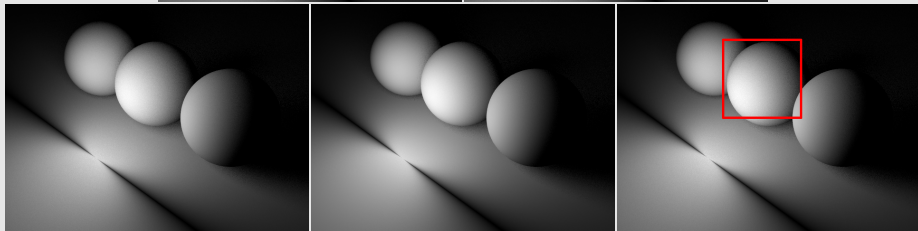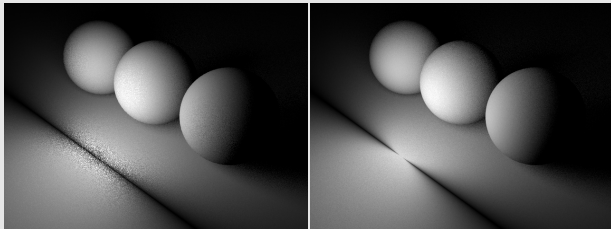
Area-preserving parameterizations for spherical ellipses

Now let's take a look on the results of our sampling, compared against area sampling and Gamito's rejection technique. In this case, we have a simple scene with a non-visible disc light source.

Mitsuba

Disk light perpendicular to ground (made invisible). Direct light renders, 16 spp.
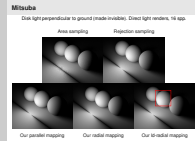

Area sampling   Rejection sampling

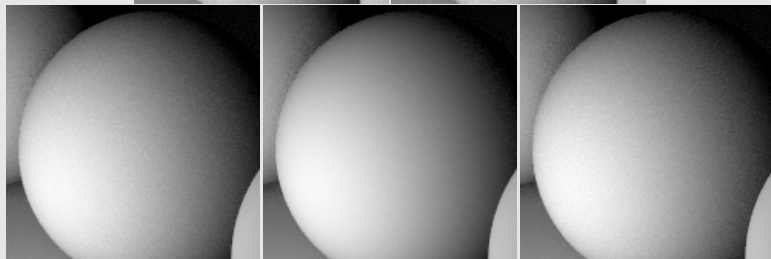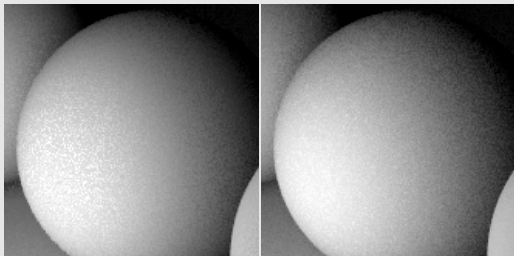Our parallel mapping   Our radial mapping   Our ld-radial mapping

39

Mitsuba
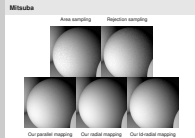
Area sampling   Rejection sampling

Our parallel mapping   Our radial mapping   Our ld-radial mapping

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Area-preserving parameterizations for spherical ellipses

The insets show the results in more detail. Note how our method gives better results than Gamito's.
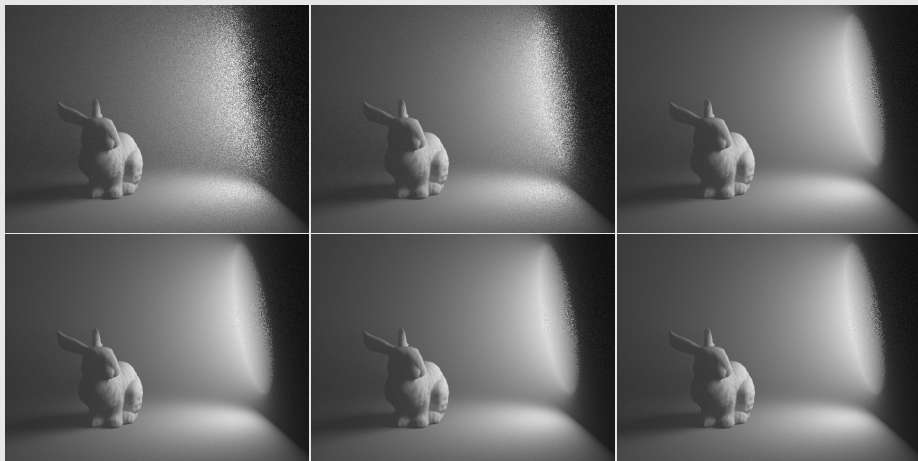
Mitsuba

Disk light embed in the media (made invisible). Direct light renders, 16 spp.

Area wrt center   Area wrt sp   Rejection sampling
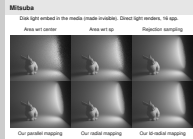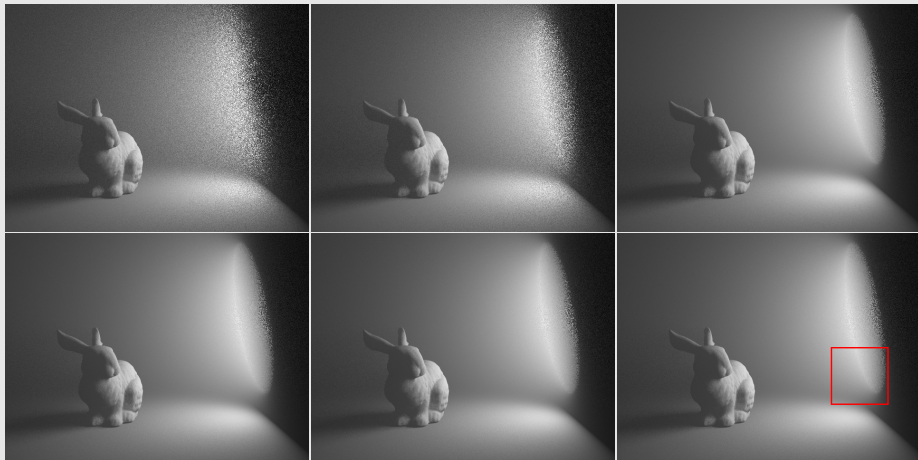

Our parallel mapping   Our radial mapping   Our ld-radial mapping

Area-preserving parameterizations for spherical ellipses

And now the same comparison in media...

Mitsuba

Disk light embed in the media (made invisible). Direct light renders, 16 spp.
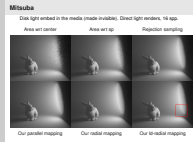
Area wrt center   Area wrt sp   Rejection sampling

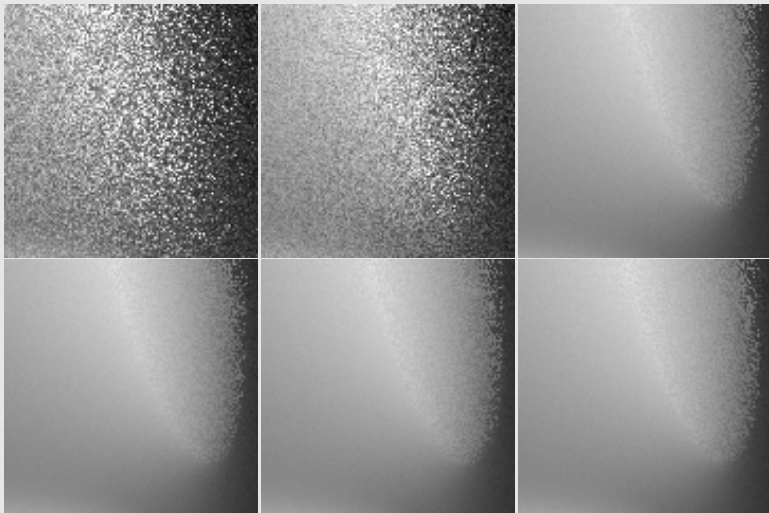Our parallel mapping   Our radial mapping   Our ld-radial mapping

Mitsuba

Area wrt center   Area wrt sp   Rejection sampling
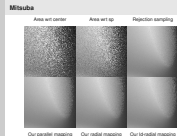
Our parallel mapping   Our radial mapping   Our ld-radial mapping
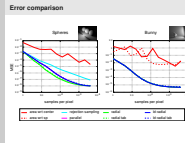
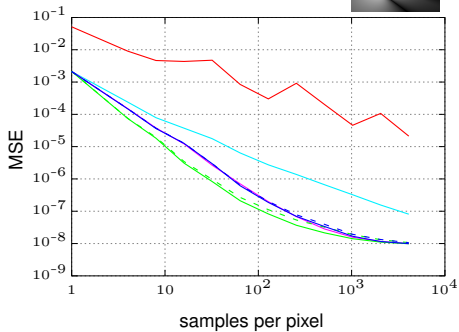Area-preserving parameterizations for spherical ellipses

Note again that specially in media we are significantly better than area sampling, while in this case Gamito's performs only slightly worse than us, due to ray sampling noise. However, as opposed to Gamito's we still keep the discrepancy of the samples.
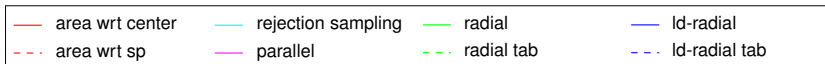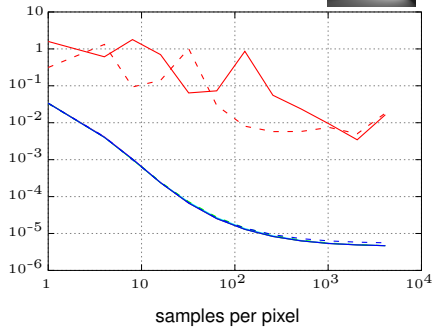
Error comparison

Spheres

Bunny

MSE / samples per pixel

Legend:
- area wrt center
- area wrt sp
- rejection sampling
- parallel
- radial
- radial tab
- ld-radial
- ld-radial tab
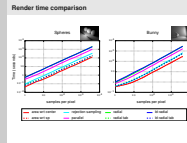
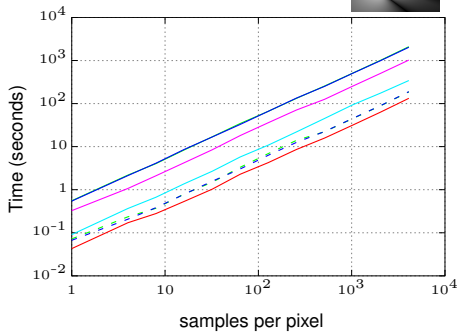Area-preserving parameterizations for spherical ellipses

Here we see the convergence curves with respect to the number of samples for both scenes.

Render time comparison

Spheres

Bunny

area wrt center — rejection sampling — radial — ld-radial
- - area wrt sp — parallel - - radial tab - - ld-radial tab
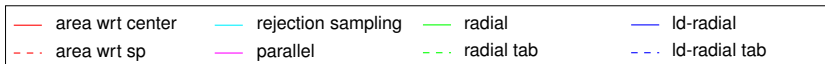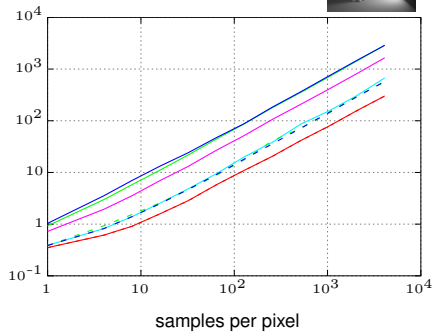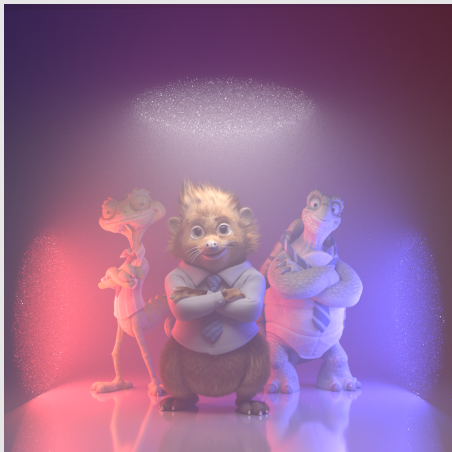
Area-preserving parameterizations for spherical ellipses

And the temporal cost. Note that these are very simple scenes, so the cost is dominated by sampling. In more realistic scenes, cost from ray tracing or shading will dominate.

Arnold

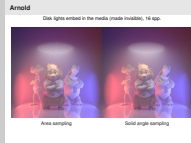Disk lights embed in the media (made invisible), 16 spp.

Area sampling   Solid angle sampling

Conclusions

- We observe that the solid angle projection of certain quadrics is a spherical ellipse.

- Using the Hat-Box theorem we can map a region on the sphere surface by mapping the bounding cylinder's lateral surface.

- Using both facts, we propose two new area-preserving mappings, which allow stratified sampling of the spherical ellipse.

  - And apply them as solid angle sampling procedures for disk area lights inside a Monte Carlo renderer.
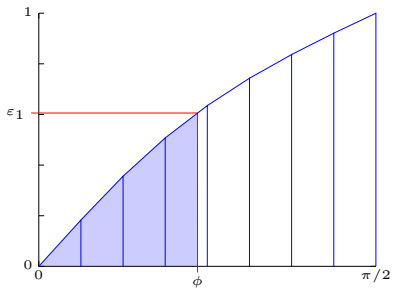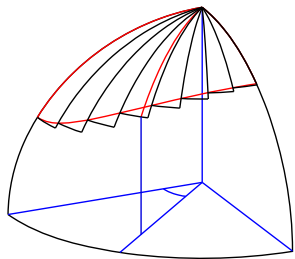
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Future work

- No analytical inversion:

  - Which forces us to use either numerical inversion or tabulation.

  - So far we have not found a spherical ellipse area expression which does not involve elliptic integrals.
  - It does not mean they don't exist (good luck with that, though)

- Include importance sampling of emission profiles, the BSDF and/or the cosine term on surfaces.

- Superellipses! Non-symmetric projection but...

Thanks!

Tabulation

Instead of using numerical inversion, we precalculate a tabulated triangle fan surrounding the spherical ellipse.

- Inverting the CDF becomes a cheap binary search.